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# New w-Convergence Conditions for the Newton-Kantorovich Method

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**Abstract.**We present new sufficient semilocal convergence conditions for the Newton-Kantorovich method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. Examples are given to show that our results apply but earlier ones do not apply to solve nonlinear equations.

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**Key Words:** Newton-Kantorovich method; Banach space; majorizing series, telescopic series.

## 1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where $F$ is a Fréchet-differentiable operator defined on an open convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations [3,7,11]. These solutions can rarely be found in closed form. That is why numerical methods are used to solve such equations.

The most popular method for generating a sequence $\{x_n\}$ approximating $x^\star$ is undoubtedly Newton-Kantorovich method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \quad (x_0 \in D). \tag{1.2}$$

Here, $F'(x) \in L(X, Y)$ the space of bounded linear operators from $X$ into $Y$ denotes the Fréchet derivative of operator $F$.

Local as well as semilocal convergence results for Newton-Kantorovich method (1.2) under various Lipschitz-type assumptions have been given by many authors [1-14].

The various semilocal convergence conditions for Newton-Kantorovich method (1.2) are only sufficient but not necessary. Hence, it is possible using the same information as before to find weaker sufficient convergence conditions. As an example (see also Section 3) we showed [3-7] that the famous Newton-Kantorovich hypothesis (see (3.1)) for solving nonlinear equations can always be replaced by an at least as weak condition (see (3.2)). Similar results for Newton-type methods have been given in [1,2,8-14]. Note that the applicability of these methods is extended, whenever weaker sufficient convergence conditions become available. Hence, such studies and results are extremely important in computational mathematics.

The affine invariant condition

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq q(\|x - y\|) \quad for\ all\ x, y \in D,$$

where, $q : [0, +\infty) \to [0, +\infty)$ is continuous, and non-decreasing has been used by many authors in the study of the Newton-Kantorovich method [5,6,7,8,9,12,13,14].

Here, we present new sufficient convergence conditions. Our results extend to solve equations

$$F(x) + G(x) = 0, \tag{1.3}$$

using

$$x_{n+1} = x_n - F'(x_n)^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0) \quad (x_0 \in D), \tag{1.4}$$

where $F$ is as above and $G : D \to Y$ is a continuous operator.

The paper is organized as follows: Section 2 contains the semilocal convergence of methods (1.2) and (1.4), whereas in Section 3 we provide special cases, and numerical examples.

## 2. Semilocal convergence

We present the semi-local convergence analysis of methods (1.2) and (1.4) in this section.

The following auxiliary result is used repeatedly in this paper.

**Lemma 1.** *(Banach lemma on invertible operators [11]) Let $T \in L(X)$. Then, $T^{-1}$ exists if and only if there is a bounded linear operator $P$ in $X$ such that $P^{-1}$ exists and*

$$\|I - PT\| < 1.$$

*If $T^{-1}$ exists, then*

$$\|T^{-1}\| \leq \frac{\|P\|}{1 - \|I - PT\|}.$$

Next, in Lemma 2, 3 and Theorem 5 we use method (1.2) to approximate a solution $x^\star$ of equation (1.1).

Let $x_0 \in D$. Suppose that the following conditions hold:

$(C_1)$ $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}\| \leq \beta$,

$$(C_2) \quad 0 < \|F'(x_0)^{-1}F(x_0)\| \leq \eta \qquad ,$$

and

$(C_3)$ there exists a continuous strictly increasing function $w : [0, +\infty) \to [0, +\infty)$ with $w^{-1} : [0, +\infty) \to [0, +\infty)$ continuous such that for all $s \geq 0, t \geq 0$ and $x, y \in D$

$$w^{-1}(s) + w^{-1}(t) \leq w^{-1}(s + t)$$

and

$$\|F'(x) - F'(y)\| \leq w(\|x - y\|).$$

It is convenient for us to define for $a_0 = b_0 = 1$, scalar sequences

$$a_{n+1} = \frac{a_n}{1 - \beta a_n w(b_n \eta)}, \tag{2.1}$$

$$c_n = \int_0^1 w(tb_n\eta)dtb_n, \tag{2.2}$$

and

$$b_{n+1} = \beta a_{n+1} c_n. \tag{2.3}$$

We provide a connection between Newton-Kantorovich method $\{x_n\}$ and scalar sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$.

**Lemma 2.** *Under the $(C_1) - (C_3)$ conditions further suppose:*

$(C_4)$ $\quad x_n \in D$

*and*

$(C_5)$ $\quad \beta a_n w(b_n\eta) < 1$ .

*Then, the following estimates hold:*

$(I_n)$ $\quad \|F'(x_n)^{-1}\| \leq a_n\beta,$

$(II_n)$ $\quad \|x_{n+1} - x_n\| = \|F'(x_n)^{-1}F(x_n)\| \leq b_n\eta,$

*and*

$(III_n)$ $\quad \|F(x_{n+1})\| \leq c_n\eta.$

*Proof.* We shall use induction to show items $(I_n) - (III_n)$. $(I_0)$ and $(II_0)$ follow immediately from the initial conditions. To show $(III_0)$, we use (1.2) for $n = 0$, $(II_0)$ and $(C_3)$ to obtain in turn

$$\begin{aligned} F(x_1) &= F(x_1) - F(x_0) - F'(x_0)(x_1 - x_0) \\ &= \textstyle\int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0)dt. \end{aligned} \tag{2.4}$$

So, we get that

$$\begin{aligned} \|F(x_1)\| &= \|\textstyle\int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0)dt\| \\ &\leq \textstyle\int_0^1 w(t\|x_1 - x_0\|)dt\|x_1 - x_0\| \leq \int_0^1 w(tb_0\eta)dtb_0\eta = c_0\eta. \end{aligned} \tag{2.5}$$

If $x_{k+1} \in D$ $(k \leq n)$, then it follows from $(C_3) - (C_5)$ and the induction hypotheses that:

$$\begin{aligned} \|F'(x_k)^{-1}\|\|F'(x_{k+1}) - F'(x_k)\| &\leq a_k\beta w(\|x_{k+1} - x_k\|) \\ &\leq \beta a_k w(b_k\eta) < 1. \end{aligned} \tag{2.6}$$

It follows from (2.6) and the Banach Lemma 1 that $F'(x_{k+1})^{-1} \in L(Y, X)$, and

$$\begin{aligned} \|F'(x_{k+1})^{-1}\| &\leq \tfrac{\|F'(x_k)^{-1}\|}{1 - \|F'(x_k)^{-1}\|\|F'(x_{k+1}) - F'(x_k)\|} \\ &\leq \tfrac{a_k\beta}{1 - \beta a_k w(b_k\eta)} = a_{k+1}\beta, \end{aligned} \tag{2.7}$$

which shows $(I_n)$ for all $n \geq 0$.

As in (2.4), we also have:

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \\ &= \textstyle\int_0^1 [F'(x_k + t(x_{k+1} - x_k)) - F'(x_k)](x_{k+1} - x_k)dt. \end{aligned} \tag{2.8}$$

Consequently, we get that

$$\begin{aligned} \|F(x_{k+1})\| &\leq \int_0^1 w(t\|x_{k+1}-x_k\|)dt\|x_{k+1}-x_k\| \\ &\leq \int_0^1 w(tb_k\eta)dtb_k\eta = c_k\eta, \end{aligned} \tag{2.9}$$

which shows $(III_n)$ for all $n \geq 0$. Moreover, by (1.2), (2.7) and (2.9) we have that

$$\begin{aligned} \|F'(x_{k+1})^{-1}F(x_{k+1})\| &\leq \|F'(x_{k+1})^{-1}\|\|F(x_{k+1})\| \\ &\leq \beta a_{k+1}c_k\eta = b_{k+1}\eta. \end{aligned} \tag{2.10}$$

That completes the induction for $(II_n)$. □ □

Next, we shall show the convergence of sequence $\{x_n\}$, which is equivalent to proving that $\{b_n\}$ is a Cauchy sequence. To this effect we need the following result:

**Lemma 3.** *Suppose:*

*Condition $(C_5)$ holds. Then, the following assertions hold:*

*(a) Scalar sequence $\{a_n\}$ increases,*

*(b) $\lim_{n\to\infty} b_n = 0$,*

*(c) $r = \sum_{k=0}^{\infty} b_k < \infty$, $b_k = \frac{1}{\eta}w^{-1}(\frac{1}{\beta}(\frac{1}{a_k} - \frac{1}{a_{k+1}}))$*

*and*

*(d) If $\overline{U}(x_0, r\eta) = \{x \in X | \|x - x_0\| \leq r\eta\} \subseteq D$, then $(C_4)$ holds.*

*Proof.* . (a) We shall show using induction that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $1-\beta a_n w(b_n\eta)$ are positive sequences. In view of the initial conditions, $a_0$, $b_0$, $c_0$ and $1-\beta a_0 w(b_0\eta)$ are positive. Assume $a_k$, $b_k$, $c_k$ and $1-\beta a_k w(b_k\eta)$ are positive for $k \leq n$. It follows from hypothesis $c_k > 0$, and (2.3) that $a_{k+1}b_{k+1} > 0$. Moreover, by (2.1), $a_{k+1} > 0$, consequently $w(b_{k+1}\eta) > 0$. Furthermore, $1 - \beta a_{k+1}w(b_{k+1}\eta) > 0$ by $(C_5)$. The induction is completed.

Solving (2.1) for $w(b_n\eta)$, we obtain

$$w(b_n\eta) = \frac{1}{\beta}(\frac{1}{a_n} - \frac{1}{a_{n+1}}). \tag{2.11}$$

By telescopic sum, we have:

$$\sum_{k=0}^{n-1} w(b_k\eta) = \frac{1}{\beta}(1 - \frac{1}{a_n}) \tag{2.12}$$

and

$$a_n = \frac{1}{1-\beta\sum_{k=0}^{n-1} w(b_k\eta)}. \tag{2.13}$$

But $1 - \beta\sum_{k=0}^{n-1} w(b_k\eta)$ decreases, so $\{a_n\}$ given by (2.13) increases. Note also that $a_n \geq a_0 = 1$.

(b) By (a), $\{a_n\}$ increases and

$$0 < \frac{1}{a_n} \leq 1. \tag{2.14}$$

Therefore, $\{\frac{1}{a_n}\}$ is monotonic on the compact set $[0, 1]$ and as such it converges to some limit denoted by $a$. By letting $n \to \infty$ in (2.11), we get

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\eta}w^{-1}(\frac{1}{\beta}(\frac{1}{a_n} - \frac{1}{a_{n+1}})) = \frac{1}{\eta}w^{-1}(\lim_{n\to\infty} \frac{1}{\beta}(\frac{1}{a_n} - \frac{1}{a_{n+1}})) = \frac{1}{\eta}w^{-1}(0) = 0. \tag{2.15}$$

(c) $r$ is finite since by the first hypothesis in $(C_3)$ and (2.15):

$$r=\sum_{k=0}^{\infty}\frac{1}{\eta}w^{-1}(\frac{1}{\beta}(\frac{1}{a_k}-\frac{1}{a_{k+1}}))\leq\frac{1}{\eta}w^{-1}(\sum_{k=0}^{\infty}\frac{1}{\beta}(\frac{1}{a_k}-\frac{1}{a_{k+1}}))=\frac{1}{\eta}w^{-1}(\frac{1}{\beta}(1-a)).$$

(d) We have $\|x_1-x_0\|\leq b_0\eta=\eta\Rightarrow x_1\in\overline{U}(x_0,r\eta)$. Assume $x_k\in\overline{U}(x_0,r\eta)\subseteq D$ for all $k\leq n$. Then, we have by Lemma 1 in turn that

$$\|x_{k+1}-x_0\|\leq\|x_{k+1}-x_k\|+\cdots+\|x_1-x_0\|\leq(b_k+\cdots+b_0)\eta\leq r\eta \tag{2.16}$$

$\Rightarrow$
$x_{k+1}\in\overline{U}(x_0,r\eta)\subseteq D.$ □ □

We can show the semilocal convergence result for Newton-Kantorovich method (1.2).

**Theorem 4.** *Under conditions $(C_1)-(C_3)$, $(C_5)$, further suppose*
$(C_6)$ $\overline{U}(x_0,r\eta)\subseteq D.$

*Then, sequence $\{x_n\}$ generated by Newton-Kantorovich method (1.2) is well defined, remains in $\overline{U}(x_0,r\eta)$ for all $n\geq 0$, and converges to a solution $x^\star\in\overline{U}(x_0,r\eta)$ of equation $F(x)=0$. Moreover, the following estimates hold:*

$$\|x_n-x^\star\|\leq\sum_{k=n}^{\infty}b_k\eta<r\eta. \tag{2.17}$$

*Furthermore, $x^\star$ is the only solution of equation $F(x)=0$ in*

$$D_1=D_0\bigcap D, \tag{2.18}$$

*where*

$$D_0=U(x_0,r_0) \tag{2.19}$$

*provided that $r_0\geq r\eta$ is the maximum number satisfying*

$$\beta\int_0^1 w((1-t)r\eta+tr_0)dt=1. \tag{2.20}$$

*Proof.* It follows from Lemmas 2 and 3 (see also $(II_n)$) that $\{x_n\}$ is a Cauchy sequence in a Banach space $X$ and as such it converges to some $x^\star\in\overline{U}(x_0,r\eta)$ (since $\overline{U}(x_0,r\eta)$ is a closed set). We have $\lim_{n\to\infty}w(b_n\eta)=0$, which implies by (2.2), the continuity of function $w$ and the assertion (b) in Lemma 3 that $\lim_{n\to\infty}c_n=0$. By letting $k\to\infty$ in (2.9) and using the continuity of operator $F$, we obtain $F(x^\star)=0$.

By $(C_6)$, we get

$$\|x_{n+1}-x_0\|\leq\sum_{k=0}^{n}\|x_{k+1}-x_k\|\leq\sum_{k=0}^{n}b_k\eta<r\eta,$$

$\Rightarrow x_{n+1}\in U(x_0,r\eta)$
$\Rightarrow x^\star=\lim_{n\to\infty}x_n\in\overline{U}(x_0,r\eta).$

Let $m>n$. Then, we have

$$\|x_n-x_m\|\leq\sum_{k=n}^{m-1}\|x_k-x_{k+1}\|\leq\sum_{k=n}^{m-1}b_k\eta<r\eta. \tag{2.21}$$

By letting $m\to\infty$ in (2.21), we obtain (2.17).

Finally, to show uniqueness, let $y^\star \in D_0$ be a solution of equation $F(x) = 0$. Define linear operator

$$M = \int_0^1 F'(x^\star + t(y^\star - x^\star))dt. \tag{2.22}$$

By $(C_1)$, $(C_3)$ and (2.20), we obtain in turn:

$$\begin{aligned} \|F'(x_0)^{-1}\|\|M - F'(x_0)\| &\leq \beta \int_0^1 w((1-t)\|x^\star - x_0\| + t\|y^\star - x_0\|)dt \\ &< \beta \int_0^1 w((1-t)r\eta + tr_0)dt = 1. \end{aligned} \tag{2.23}$$

It follows from (2.23), and the Banach lemma 2.1 that $M^{-1}$ exists. Using the identity

$$0 = F(y^\star) - F(x^\star) = M(y^\star - x^\star),$$

we deduce $x^\star = y^\star$. □ □

*Remark* 5. It follows from $(C_3)$ that

$(C_3)'$ there always exists a continuous non-decreasing function $w_0 : [0, +\infty) \to [0, +\infty)$ with $w_0(0) = 0$ such that for all $x \in D$

$$\|F'(x) - F'(x_0)\| \leq w_0(\|x - x_0\|).$$

Note that

$$w_0 \leq w$$

holds in general, and $\frac{w}{w_0}$ can be arbitrarily large [3-7]. Hence $(C_3)'$ is not an additional hypothesis. In view of $(C_3)'$, and (2.5) $c_0$, and $a_1$ can be defined in a tighter way by

$$a_1 = \frac{a_0}{1 - \beta a_0 w_0(b_0\eta)}$$

and

$$c_0 = \int_0^1 w_0(tb_0\eta)dtb_0.$$

The new $\{a_n\}, \{b_n\}$ and $\{c_n\}$ sequences are tighter majorizing (for $\{x_n\}$) than before under the same computational cost. Moreover, the uniqueness ball is extended, since $w_0$ can replace $w$ (see (2.23)) in condition (2.20).

The results obtained here can be extended to hold for equations containing a not necessarily differentiable term.

In the remaining results we use method (1.4) to approximate a solution $x^\star$ of equation (1.3).

Let us suppose:

$(C_7)$ there exists a continuous, non-decreasing function $v_0 : [0, +\infty) \to [0, +\infty)$ with $v(0) = 0$ such that for all $x, y \in D$:

$$\|G(x) - G(y)\| \leq v(\|x - y\|)\|x - y\|.$$

Define sequence $\{c_n\}$ by

$$c_n = [\int_0^1 w(tb_n\eta)dt + v(b_n\eta)]b_n,$$

where as $\{a_n\}$ and $\{b_n\}$ are given by (2.1) and (2.3), respectively.

Then, using the identity:

$$\begin{aligned} F(x_{n+1}) + G(x_{n+1}) &= \int_0^1 [F'(x_n + t(x_{n+1} - x_n)) - F'(x_n)](x_{n+1} - x_n)dt \\ &\quad + G(x_{n+1}) - G(x_n), \end{aligned} \tag{2.24}$$

(instead of (2.4) (for $n = 0$), (2.8)) and following the rest of the proof of Theorem 4 (excluding the uniqueness part) we arrive at:

**Theorem 6.** *Under the conditions $(C_1) - (C_3)$, $(C_5) - (C_7)$ the following hold:*

*Sequence $\{x_n\}$ generated by Newton-Kantoroovich method (1.4), is well defined, remains in $\overline{U}(x_0, r\eta)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, r\eta)$ of equation $F(x) + G(x) = 0$. Moreover, the following estimates hold:*

$$\|x_n - x^\star\| \leq \sum_{k=n}^{\infty} b_k \eta < r\eta. \tag{2.25}$$

*We can show a uniqueness result but we use a condition other than (2.20).*

**Proposition 7.** *Under the hypotheses of Theorem 2.6, further suppose:*

*there exists $r_1 \geq r\eta$ such that*

$$\beta(\int_0^1 w(tr_1)dt + v(r_1)) \leq aq < 1 \quad for\ some\ q \in (0,1), \tag{2.26}$$

*then $x^\star$ is the unique solution of equation $F(x) + G(x) = 0$ in $D_3 = D \bigcap D_2$, where*

$$D_2 = \overline{U}(x_0, r_1),$$

*and $a$ is given in Lemma 2.3.*

*Proof.* Let $y^\star \in D_3$ be a solution of equation $F(x) + G(x) = 0$. Using (1.4), we get the identity

$$\begin{aligned} x_{n+1} - y^\star &= -F'(x_n)^{-1}[F(x_n) - F(y^\star) - F'(x_n)(x_n - y^\star) + G(x_n) - G(y^\star)] \\ &= -F'(x_n)^{-1}[\textstyle\int_0^1 \big(F'(y^\star + t(x_n - y^\star)) - F'(x_n)\big)dt(x_n - y^\star) \\ &\quad +G(x_n) - G(y^\star)], \end{aligned} \tag{2.27}$$

so,

$$\begin{aligned} \|x_{n+1} - y^\star\| &\leq a_n\beta[\textstyle\int_0^1 w(t\|x_n - y^\star\|)dt + v(\|x_n - y^\star\|)]\|x_n - y^\star\| \\ &\leq a_n\beta[\textstyle\int_0^1 w(tr_1)dt + v(r_1)]\|x_n - y^\star\| \\ &\leq q\|x_n - y^\star\|. \end{aligned} \tag{2.28}$$

Hence, we get

$$\|x_{n+1} - y^\star\| \leq q^n\|x_0 - y^\star\| \leq q^n r_1,$$

which implies $\lim_{n\to\infty} x_n = y^\star$. But we know that $\lim_{n\to\infty} x_n = x^\star$. Hence, we deduce $x^\star = y^\star$. ☐

It turns out that condition $(C_5)$ can be replaced by the at least as weak

$(C_5)'$ $\quad \beta w_0(d_n\eta) < 1$.

Indeed, introduce scalar sequences $\{p_n\}$, $\{d_n\}$ by

$$p_0 = \beta, \quad d_0 = 1,$$

$$d_n = \sum_{k=0}^{n} b_k$$

and

$$p_{n+1} = \frac{p_0}{1 - \beta w_0(d_n\eta)}.$$

We get

$$d_n = \frac{1}{\eta} w_0^{-1}(\frac{1}{p_0} - \frac{1}{p_{n+1}}).$$

Then, in view of the observation

$$\begin{aligned} & \|F'(x_0)^{-1}\|\|F'(x_{k+1}) - F'(x_0)\| \\ \leq\ & \beta w_0(\|x_{k+1} - x_0\|) \\ \leq\ & \beta w_0(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\|) \\ \leq\ & \beta w_0((b_k + b_{k-1} + \cdots + b_0)\eta) = \beta w_0(d_k\eta), \end{aligned} \tag{2.29}$$

estimate (2.7) can be replaced by the at least as precise

$$\begin{aligned} \|F'(x_{k+1})^{-1}\| &\leq \frac{p_0}{1 - \beta w_0(\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\|)} \\ &\leq \frac{p_0}{1 - \beta w_0(d_k\eta)}. \end{aligned} \tag{2.30}$$

With these changes, we arrive at the following analogs of Lemmas 2, 3, Theorem 4, 6 and Proposition 7. □

**Lemma 8.** *Suppose $(C_1) - (C_4)$, $(C_3)'$ and $(C_5)'$ hold. Then, the following estimates hold*

$$\|F'(x_n)^{-1}\| \leq p_n,$$
$$\|x_{n+1} - x_n\| \leq b_n\eta$$

*and*

$$\|F(x_{n+1})\| \leq c_n\eta.$$

**Lemma 9.** *Suppose $(C_5)'$ holds. Then, sequence*

*(a) $\{p_n\}$ increases,*
*(b) $\{d_n\}$ is increasingly convergent,*
*(c) $\lim_{n\to\infty} b_n = 0$,*
*(d) $r = \sum_{k=0}^{\infty} b_k < \infty$*
*and*
*(e) If $\overline{U}(x_0, r\eta) \subseteq D$, then $(C_4)$ holds.*

Similarly, we obtain analogs of Theorem 4,6 and Proposition 7 (simply replace $(C_5)$ by $(C_5)'$)).

*Remark* 10. The results obtained here can further be refined, if we further assume:
$(C_3)''$ there exists a function $p : [0,1] \to [0,+\infty)$ such that

$$w(st) \leq p(s)w(t) \quad for\ \ all\ \ s \in [0,1]\ \ and\ \ t \in [0,+\infty).$$

This condition has been successfully used to sharpen the error bounds for particular expressions [5,6,7,8,9,12,13,14]. Note that such a function $p$ always exists. Indeed, if $w$ is a nonzero function on $\mathbb{R}_+$, then one can define $p : [0,1] \to [0,+\infty)$ by

$$p(s) = \sup\{\frac{w(st)}{w(t)} : t \in [0,+\infty), with\ \ w(t) > 0\}.$$

Note that in this case the results obtained in this study hold with $\int_0^1 w(tb_n\eta)dt$ replaced by $Pw(b_n\eta)$, where

$$P = \int_0^1 p(s)ds.$$

Finally, note that the results obtained here can be provided in affine invariant form, if we replace operator $F$ by $F'(x_0)^{-1}F$ [5-7].

## 3. Special cases and applications

Condition $(C_5)$ is difficult to verify in general. However, $(C_5)$ holds in some very interesting cases. Let us consider the Lipschitz case, i.e., $w(s) = Ls$, $w_0(s) = L_0 s$, and $G = 0$. Then, the famous for its simplicity and clarity Newton-Kantorovich hypothesis

$$h_K = \beta L \eta \leq \frac{1}{2} \tag{3.1}$$

implies condition $(C_5)$ and

$$r_K = \frac{2}{1+\sqrt{1-2h_K}} \quad [11].$$

Moreover, our condition given in [3], [11] by

$$h_{AH} = \beta \overline{L} \eta \leq \frac{1}{2}, \tag{3.2}$$

where,

$$\overline{L} = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0 L})$$

also implies $(C_5)$ and

$$r_{AH} = \frac{2}{2-\alpha\beta},$$

where,

$$\alpha = \frac{4L}{L+\sqrt{L^2+8L_0L}}.$$

Note that

$$h_K \leq \frac{1}{2} \rightarrow \quad h_{AH} \leq \frac{1}{2}$$

but not necessarily vise versa unless if $L_0 = L$.

In the first example we show that the Kantorovich hypothesis (see (3.1)) is satisfied with the bigger uniqueness ball of solution than before [2], [7].

**Example 11.** *Let $X = Y = \mathbb{R}$ be equipped with the max-norm. Let $x_0 = 1$, $D = U(x_0, 1-q)$, $q \in [0,1)$ and define function $F$ on $D$ by*

$$F(x) = x^3 - q. \tag{3.3}$$

*Then, we obtain that $\beta = \frac{1}{3}$, $L = 6(2-q)$, $L_0 = 3(3-q)$ and $\eta = \frac{1}{3}(1-q)$. Then, the famous for its simplicity and clarity Kantorovich hypothesis for solving equations using (NKM) [1,2,7] is satisfied, say for $q = .6$, since*

$$h_K = \beta L \eta = \frac{2}{3}(2-q)(1-q) = 0.373333\ldots < \frac{1}{2}. \tag{3.4}$$

*Hence, (NKM) converges starting at $x_0 = 1$. We also have that $r = 1.330386708$, $\eta = .133\ldots$, $r\eta = .177384894$, $L_0 = 7.2 < L = 8.4$ and $r_0 = \frac{2}{\beta L_0} - r\eta = .655948439$. That is our Theorem 2.4 guarantees the convergence of (NKM) to $x^\star = \sqrt[3]{0.6} = .843432665$ and the uniqueness ball is better than the one given in (KT).*

In the second example we apply Theorem 4 to a nonlinear integral equation of Chandrasekhar-type.

**Example 12.** *Let us consider the equation*

$$x(s) = 1 + \frac{s}{4}x(s)\int_0^1 \frac{x(t)}{s+t}dt, \quad s \in [0,1]. \tag{3.5}$$

*Note that solving (3.5) is equivalent to solving $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ defined by*

$$[F(x)](s) = x(s) - 1 - \frac{s}{4}x(s)\int_0^1 \frac{x(t)}{s+t}dt, \quad s \in [0,1]. \tag{3.6}$$

*Using (3.6), we obtain that the Fréchet-derivative of $F$ is given by*

$$[F'(x)y](s) = y(s) - \frac{s}{4}y(s)\int_0^1 \frac{x(t)}{s+t}dt - \frac{s}{4}x(s)\int_0^1 \frac{y(t)}{s+t}dt, \quad s \in [0,1]. \tag{3.7}$$

*Let us choose the initial point $x_0(s) = 1$ for each $s \in [0,1]$. Then, we have that $\beta = 1.534463572$, $\eta = .2659022747$, $L_0 = L = \ln 2 = .693147181$, $h = .392066334$ and $r = 1.23784269$ (see also [1,2,3,7]). Then, hypotheses of Theorem 2.4 are satisfied. In consequence, equation $F(x) = 0$ has a solution $x^\star$ in $U(1,\rho)$, where $\rho = r\eta = .298816793$.*

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